ADAPTIVE FILTERING AND PERIODIC INPUTS: CONDITIONS FOR AN EXPONENTIAL DIVERGENCE.

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ABSTRACT
Adaptive filters find applications in many areas of signal processing and echo cancellation is one of them. It has been observed that, due to non-linearities inherent in the implementation, echo cancelers can diverge when their input is periodic. In this paper we examine the behaviour of adaptive filters in the presence of periodic inputs and provide a theoretical explanation and conditions for a potential exponential divergence.

1. INTRODUCTION
Adaptive filters find application in many areas of signal processing including echo cancelling, adaptive equalisation, and spectrum estimation. A popular algorithm, which has been extensively used, is the adaptive Least-Mean-Square (LMS). Widrow and others have studied its various properties and applications for close to three decades [1][2][3]. This algorithm utilizes a nonrecursive filter structure (FIR) driven by a primary input \( x[n] \) (Fig. 1a). The filter weights are updated iteratively based upon the difference, \( e[n] \), between the filter output, \( \hat{y}[n] \), and a reference input, \( y[n] \), so as to minimize the mean square of this difference. Simplicity and relatively good performance are the main advantages of the above filter.

During the past years the performance of the LMS algorithm has been studied mainly for stochastic inputs and a variety of interesting properties and theorems have been derived. However, in the real world, periodic signals occupy a significant portion of the signals that we encounter and assumptions of white noise, uncorrelated data, etc., are often violated. Also moving from the mathematical model of LMS to a real silicon realization we have to sacrifice some features, like infinite precision arithmetic, instantaneous update, as many taps as you want, etc.

In this work we will analyse adaptive filters under periodic input and we shall

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Figure 1. A FIR adaptive system.
present some potential problems that can be developed from the deviations between the silicon algorithm and the ideal mathematical algorithm.

2. THE MODEL

Denoting by $h[n,k]$ the $k$-th coefficient of an $M$-th order LMS filter at time $n$, (2). This is illustrated in Fig. 2 where we assume that the signal used for deriving $e[n]$ in the output equation (2) is a non-linear transformation (NL1) of $x[n]$ hereafter denoted with $x_{CV}[n]$, while the one used for the update equation is another non-linear transformation (NL2) of $x[n]$, hereafter denoted with $x_{CR}[n]$ (CV stands

![Figure 2. An adaptive filter using the LMS algorithm with different non-linearities in the convolution and in the correlation multipliers.](image)

the update equation is given by

$$h[n+1,k] = h[n,k] + \mu e[n]x[n-k]$$  \hspace{1cm} (1)

where $\mu$ is the adaptation constant, $e[n]$ is the error that we want to minimize and $x[n]$ denotes the complex conjugate of $x[n]$.

We shall assume that $x[n]$ is periodic with period $N$. In order to simplify our analysis we shall also assume, with some loss of generality, that the reference input $y[n]$ is zero. Then

$$e[n] = -h[n,k] \star x[n]$$  \hspace{1cm} (2)

where $\star$ denotes convolution.

Due to different nonlinearities, time variable gain’s etc., we shall assume that the input signal $x[n]$ in equation (1) is different than the one used in equation for Convolution and CR stands for Correlation). One case, for example, where this situation occurs is when we use different quantizations for the $x[n]$ present in the update equation (1), and the one present in the output equation (2). With these assumption the update (correlation) equation becomes

$$h[n+1,k] = h[n,k] + \mu e[n]x_{CR}[n-k]$$  \hspace{1cm} (3)

with the additional constrain that the positive number $\mu$ is constant over $k$.

The output (convolution) equation also becomes

$$e[n] = -h[n,k] \star x_{CV}[k]$$  \hspace{1cm} (4)

Since $x[n]$ was assumed to be periodic with period $N$, $x_{CR}[n]$ and $x_{CV}[n]$ are also
periodic sequences with the same period. Then
\[ x_{CR}[n] = \sum_{m=0}^{N-1} A_m w^{mn} \quad w = e^{2\pi j/N} \]
(5)
where \( A_m \) are the Discrete Fourier Series coefficients of \( x_{CR}[n] \) and
\[ x_{CV}[n] = \sum_{m=0}^{N-1} B_m w^{mn} \quad w = e^{2\pi j/N} \]
(6)
with \( B_m \) \( m=0,1,2,...,N-1 \) the DFS coefficients of \( x_{CV}[n] \). Denoting by
\[ \mathbf{H}[n] = [h[n,0], h[n,1], \ldots, h[n,M-1]]^T \]
\[ \mathbf{X}_{CR}[n] = [x_{CR}[n], x_{CR}[n-1], \ldots, x_{CR}[n-M+1]]^T \]
\[ \mathbf{X}_{CV}[n] = [x_{CV}[n], x_{CV}[n-1], \ldots, x_{CV}[n-M+1]]^T \]
we can rewrite relation (3) in a matrix form as
\[ \mathbf{H}[n+1] = \mathbf{H}[n] - \mu_0 \mathbf{X}_{CR}[n] \mathbf{X}_{CV}[n] \mathbf{H}[n] \]
(8)
Since \( \mathbf{X}_{CR}[n] \) and \( \mathbf{X}_{CV}[n] \) are periodic we can also express them in terms of the DFS coefficients of \( x_{CR}[n] \) and \( x_{CV}[n] \). In particular, we have (see (4)),
\[ \mathbf{X}_{CR}[n] = \mathbf{F}[n] \mathbf{A} \]
\[ \mathbf{X}_{CV}[n] = \mathbf{F}[n] \mathbf{B} \]
where
\[ \mathbf{A} = [A_0, A_1, \ldots, A_{N-1}]^T \]
\[ \mathbf{B} = [B_0, B_1, \ldots, B_{N-1}]^T \]
and \( \mathbf{F}[n] \) (the DFS transformation matrix) is an \( \mathbb{M} \times \mathbb{N} \) matrix given by
\[
\mathbf{F}[n] = \begin{bmatrix}
1 & w^n & w^{2n} & \cdots & w^{(N-1)n} \\
1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(N-1)(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{(n-M+1)} & w^{2(n-M+1)} & \cdots & w^{(N-1)(n-M+1)}
\end{bmatrix}
\]
(11)
Thus (8) takes the form
\[ \mathbf{H}[n+1] = \mathbf{H}[n] - \mu_0 \mathbf{F}[n] \mathbf{A}^* \mathbf{B}^T \mathbf{F}[n] \mathbf{H}[n] \]
\[ = \mathbf{I}_{\mathbb{M} \times \mathbb{M}} - \mu_0 \mathbf{F}[n] \mathbf{A}^* \mathbf{B}^T \mathbf{F}[n] \mathbf{H}[n] \]
\[ = \mathbf{F}[n] \left( \frac{1}{\mathbb{N}} \mathbf{I}_{\mathbb{N} \times \mathbb{N}} - \mu_0 \mathbf{A}^* \mathbf{B}^T \right) \mathbf{F}[n] \mathbf{H}[n] \]
(12)
where the last equality is obtained from the fact that
\[ \mathbf{F}[n] \mathbf{I}_{\mathbb{N} \times \mathbb{N}} \mathbf{F}[n]^T = \mathbf{M}_{\mathbb{N} \times \mathbb{M}} \]
Noticing also that
\[ \mathbf{F}[n+1] = \mathbf{F}[n] \mathbf{W} = \mathbf{F}[0] \mathbf{W}^{n-1} \]
(14)
where
\[ \mathbf{W} = \text{diag}\{1, w, w^2, \ldots, w^{N-1}\} \]
(15)
and defining as
\[ \mathbf{N} \mathbf{H}_F[n] = \mathbf{F}_{\mathbb{M} \times \mathbb{N}}^T \mathbf{H}[n] \]
(16)
where \( H[n] \) denotes the DFS of \( h[n] \) we obtain from (12)
\[ \mathbf{H}[n+1] = \mathbf{W}^{n+1} \mathbf{N} \mathbf{H}_F[n] = \mathbf{F}_{\mathbb{N} \times \mathbb{N}}^T \mathbf{H}[n+1] \]
\[ = \mathbf{W} \mathbf{F}_{\mathbb{N} \times \mathbb{N}}^T \mathbf{F}_{\mathbb{N} \times \mathbb{N}} \frac{1}{\mathbb{N}} \mathbf{G} \mathbf{F}_{\mathbb{N} \times \mathbb{N}}^T \mathbf{H}[n] \]
\[ = \frac{1}{\mathbb{N}} \mathbf{W} \mathbf{F}_{\mathbb{N} \times \mathbb{N}}^T \mathbf{F}_{\mathbb{N} \times \mathbb{N}} \mathbf{G} \mathbf{H}[n] \]
\[ = \frac{1}{\mathbb{N}} \mathbf{W}^{n+1} \mathbf{F}_{\mathbb{N} \times \mathbb{N}}^T \mathbf{F}_{\mathbb{N} \times \mathbb{N}} \mathbf{W}^{-n} \mathbf{G} \mathbf{H}[n] \]
with
\[ \mathbf{G} = \mathbf{I} - \mu_0 \mathbf{N} \mathbf{A}^* \mathbf{B}^T \]
(18)
If \( \mathbb{M} = \mathbb{N} \), then equation (17) becomes
\[ \mathbf{H}[n+1] = \mathbf{W} \mathbf{G} \mathbf{H}[n] \]
i.e. when \( \mathbb{M} = \mathbb{N} \), the adaptive system is time invariant in terms of the transformed variables \( H[n] \), hence its stability can be analyzed with the best known techniques for a time invariant system.
Equations (12) and (17) are the basic equations for our model.

3. THEORETICAL RESULTS
We believe that the following statement is true

\[
\text{If } |1 - \mu_0 \mathbb{N} \mathbf{A}^* \mathbf{B}| > 1 \text{ for any } k \text{ then the system is unstable.}
\]

In a more engineering terms, the above relation states that, if by some mechanism, for example nonlinearities, time varying gain or anything else, we generate harmonics in \( x_{CR}[n] \) which are out of phase with the harmonics present in \( x_{CV}[n] \), then equation (3) will diverge (exponentially). This is intuitively expected, because for those harmonics, equation (3) will have a negative adaptation gain (positive feedback) and in every step will move the impulse response into the wrong direction.
However, we were able to prove only the following somewhat weaker results.
Case 1. M=N

In this case the length, M, of the canceler is an integer multiple of the period, N, of the input signal x[n].

**THEOREM 1:** If

\[ 1 - \mu N \sum_{k=1}^{N} A_k^* B_k > 1 \]  \hspace{2cm} (19)

the system is unstable.

Note. If \( A_k = B_k \) which is the usual case where the signals \( x_{ce}[n] \) and \( x_{cn}[n] \) are assumed to be the same, then equation (19) becomes the well known stability condition

\[ \mu_0 < \frac{1}{N \sum_{k=1}^{N} |x[n]|^2} \]  \hspace{2cm} (20)

**THEOREM 2:** For small values of \( \mu_0 \) \( (\mu_0 << 1) \), if for any \( k \)

\[ |1 - \mu_0 N A_k^* B_k| > 1 \]  \hspace{2cm} (21)

then the system is unstable (exponentially). In a more mathematical way, we proved that if \( \lambda_0 \) are the eigenvalues of (17), then

\[ \lim_{\mu \to 0^+} |\lambda_1| = |1 - \mu_0 N A_k^* B_k| \]

Hereafter, when we say a statement is true for small values of \( \mu_0 \) we mean that the statement is true for \( \mu_0 \to 0^+ \).

It is clear that if \( A_k \) and \( B_k \) are 180 out of phase then for any positive value of \( \mu_0 \) the LMS algorithm diverges.

Case 2: M<N.

**THEOREM 3:** For small values of \( \mu_0 \) the system is stable if for every \( k \)

\[ |1 - \mu_0 N^2 A_k^* B_k| < 1 \]  \hspace{2cm} (22)

**THEOREM 4:** For small values of \( \mu_0 \) \( (\mu_0 << 1) \) and assuming that the number of non-zero harmonics \( A_k^* B_k \) is less than \( M \), and that \( A_k^* B_k \) is real, the system is exponentially unstable if

\[ |1 - \mu_0 N^2 A_k^* B_k| > 1 \]  \hspace{2cm} (23)

or equivalently \( A_k^* B_k < 0 \).

If the number of non-zero harmonics is less than \( M \), the order of the LMS algorithm, then there are also other mechanisms [4], generating divergence but none of them is of exponential type.

Those kinds of divergence can be removed by introducing a leak [5] in the coefficients of the adaptive filter, i.e.,

\[ r \mathbf{H}[n] - \mu_0 x^*[n]\mathbf{X^T}_{cn}[n] \mathbf{H}[n] \]  \hspace{2cm} (24)

4. CONCLUSION

It has been observed that in the presence of periodic signals echo cancelers some times diverge. In this work we have presented a model and an analysis that we believe explains the observed divergence. Numerical simulation and experimental work appear to agree with our findings.

REFERENCES


