IMPROVEMENT OF RANGE RESOLUTION BY SPECTRAL EXTRAPOLATION

A. Papoulis¹ and C. Chamzas

Polytechnic Institute of New York
Department of Electrical Engineering
Farmingdale, New York 11735

Under various simplifying assumptions, the reflected signal \( y(t) \) in the interrogation of a substance by an ultrasonic wave is a convolution of the transmitted signal \( x(t) \) with a function \( h(t) \) that is related to the reflection coefficient of the medium in the direction of propagation. The function \( h(t) \) can, in principle, be determined by deconvolution. However, since the band \( B \) of the spectrum \( X(\omega) \) of \( x(t) \) is finite, the frequency components of \( h(t) \) outside \( B \) cannot be found reliably. In this paper, a method is presented for extrapolating \( X(\omega) \) beyond \( B \). The resulting increase in resolution is limited only by the level of noise. The method is particularly effective if \( h(t) \) is a sum of impulses.

Key words: Deconvolution; diagnostics; echoes; extrapolation; resolution; ultrasonic.

1. Introduction

Ultrasonic waves are used in metallurgy, in medicine, and in other areas to determine the structure of a medium. The medium is interrogated by a narrow beam (fig. 1) and the reflected signal \( y(t) \) is used to determine various properties of the medium, in particular, the location of its surface of discontinuity. Under various simplifying assumptions, the signal \( y(t) \) can be expressed as a convolution integral:

\[
y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau
\]

(1)

where \( x(t) \) is the transmitted signal (fig. 2a) and \( h(t) \) is a function related to the reflection coefficients of the medium in the direction of propagation. The variable \( t \) is proportional to the distance along the beam. The assumptions leading to eq. (1) and the relationship between \( h(t) \) and the parameters of the medium will not be considered here.

If the medium consists of homogeneous layers, then \( h(t) \) is a sum of impulses as in figure 2b:
Fig. 1  Schematic of an ultrasonic system

Fig. 2  (a) Transmitted signal $x(t)$
(b) Impulse response $h(t)$ of idealized medium
(c) Reflected signal $y(t)$ for $d = 0.375T$
(d) Reflected signal $y(t)$ for $d = 1.25T$
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\[ h(t) = \sum_i c_i \delta(t-t_i) \quad (2) \]

where the points \( t_i \) correspond to the locations of the separation surfaces, and the coefficients \( c_i \) are related to the reflection coefficients at these surfaces. In this case, eq. (1) yields

\[ y(t) = \sum_i c_i x(t-t_i) \quad (3) \]

and the problem is to determine the unknowns \( c_i \) and \( t_i \) in terms of the observed signal \( y(t) \).

If \( x(t) \) is a pulse whose duration \( d \) is smaller than the minimum distance \( t_i-t_j \) between neighboring impulses, then the various terms in eq. (3) do not overlap (fig. 2c), hence, the constants \( c_i \) and \( t_i \) can be readily found. This is not the case, however, if \( d \) is larger than \( t_i-t_j \) (fig. 2d). The resolution of the system, i.e., the smallest distance between reflecting surfaces that can be detected without special processing is thus, proportional to the duration \( d \) of \( x(t) \). To improve the resolution, we must decrease \( d \), or, equivalently, we must design a transducer with large bandwidth. In this paper, we present a method for increasing the resolution of a given system by numerical processing of the signal \( y(t) \).

In principle, \( h(t) \) can be determined simply in terms of \( x(t) \) and \( y(t) \) by deconvolution. Indeed, taking Fourier transforms of both sides of eq. (1), we conclude from the convolution theorem that

\[ H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad (4) \]

This yields the Fourier transform \( H(\omega) \) of the unknown \( h(t) \) in terms of the Fourier transforms \( X(\omega) \) and \( Y(\omega) \) of the signals \( x(t) \) and \( y(t) \) respectively. To find \( h(t) \), it suffices, therefore, to compute the inverse transform of the ratio \( Y(\omega)/X(\omega) \).

However, the resulting values of \( H(\omega) \) in the region of the frequency axis where \( X(\omega) \) is of the order of the background noise are not reliable. For a typical transducer, \( X(\omega) \) is a curve as in figure 3 taking significant values in the band \((\omega_1, \omega_2)\) only; hence, \( H(\omega) \) can be determined reliably only in this band. Our problem, therefore, is to extrapolate \( H(\omega) \) for \( \omega < \omega_1 \) and \( \omega > \omega_2 \). We shall do so under the assumption that \( h(t) \) is a sum of impulses as in eq. (2). We note that, since \( X(\omega) \) is not strictly bandlimited, the frequencies \( \omega_1 \) and \( \omega_2 \) are rather arbitrary. If they are so chosen that \( X(\omega) \) is significantly larger than the noise level, then the error in the estimation of \( H(\omega) \) by the ratio \( Y(\omega)/X(\omega) \) is small. However, the extrapolation problem is then more difficult because the length \( \omega_2-\omega_1 \) of the resulting interval in which \( H(\omega) \) is assumed known is small.

We note, finally, that in the presence of noise, the best estimate of \( H(\omega) \) in the interval \((\omega_1, \omega_2)\) is not the ratio \( Y(\omega)/X(\omega) \). It
can be shown \[4\] that the estimate of \(H(\omega)\) in the vicinity of the low values of \(X(\omega)\) can be improved if the properties of the noise are known.

The proposed method of determining \(h(t)\) in terms of only a segment of its Fourier transform \(H(\omega)\), is a nonlinear adaptive modification of an extrapolation method developed in reference \[2\]. In the next section, we review briefly the relevant parts of this paper.

2. Spectral extrapolation of time-limited signals

Consider a function \(h(t)\) with Fourier transform \(H(\omega)\). We assume that only the segment

\[
W_1(\omega) = \begin{cases} 
H(\omega) & \omega \in B \\
0 & \omega \in \overline{B}
\end{cases}
\]

(5)

of \(H(\omega)\) is known, where \(B\) is a certain band on the frequency axis and \(\overline{B}\) is its complement. In general, \(h(t)\) cannot be determined in terms of \(W_1(\omega)\). However, if \(h(t)\) is time-limited, i.e., if

\[
h(t) = 0 \quad \text{for } |t| > T
\]

(6)

then \[2\] it can be uniquely determined in terms of \(W_1(\omega)\).

In the ultrasonics problem, the duration \(2T\) of \(h(t)\) is proportional to the thickness of the medium, and the band \(B\) is the interval \((\omega_1, \omega_2)\) in which \(X(\omega)\) takes significant values. The function \(W_1(\omega)\) equals the ratio \(Y(\omega)/X(\omega)\) for every \(\omega\) in \(B\).
Using a numerical iteration involving only FFT’s, we shall determine \( H(\omega) \) for every \( \omega \).

**First step.** We compute the inverse transform

\[
\begin{align*}
w_1(t) &= \frac{1}{2\pi} \int_{-B}^{B} W_1(\omega) e^{j\omega t} d\omega \\
\end{align*}
\]  

(7)

of \( W_1(\omega) \). We form the function

\[
h_1(t) = \begin{cases} 
w_1(t) & |t| < T \\
0 & |t| > T 
\end{cases}
\]

(8)

obtained by truncating \( w_1(t) \) for \( |t| > T \) (fig. 4). We compute the Fourier transform

\[
H_1(\omega) = \int_{-T}^{T} h_1(t) e^{-j\omega t} dt
\]

(9)

of \( h_1(t) \), and form the function

\[
W_2(\omega) = \begin{cases} 
W_1(\omega) = H(\omega) & \omega \in B \\
H_1(\omega) & \omega \in \overline{B}
\end{cases}
\]

(10)

**nth iteration.** We compute the inverse transform

\[
w_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_n(\omega) e^{j\omega t} d\omega
\]

(11)

of the function \( W_n(\omega) \) determined at the end of the \( n-1 \) iteration. We form the function

\[
h_n(t) = \begin{cases} 
w_n(t) & |t| < T \\
0 & |t| > T 
\end{cases}
\]

(12)

obtained by truncating \( w_n(t) \) for \( |t| > T \) (fig. 5). We compute the Fourier transform

\[
H_n(\omega) = \int_{-T}^{T} h_n(t) e^{-j\omega t} dt
\]

(13)

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Fig. 4. Description of first iteration Upper Level; time signals; lower level; their spectra. Arrows indicate the order of the various computations starting with $W_1(\omega)$. 
Fig. 5 Description of \( n \)th iteration

of \( h_n(t) \), and form the function

\[
W_{n+1}(\omega) = P(\omega)H_n(\omega) + \left[1 - P(\omega)\right]H_n(\omega)
\]

(14)

where \( P(\omega) \) is a rectangular window:

\[
P(\omega) = \begin{cases} 
1 & \omega \in B \\
0 & \omega \in \overline{B} 
\end{cases}
\]

(15)

as in figure 5.

We have shown [2, 3] that, in the absence of noise, the sequence \( h_n(t) \) so formed tends to the unknown signal \( h(t) \) as \( n \) tends to infinity. The effects of noise are considered in reference [2].

In eq. (14), we used for the term \( P(\omega) \) a rectangular window. This, however, is not necessary [5, 6]. It can be shown that the iteration converges for any \( P(\omega) \) provided that

\[
0 \leq P(\omega) \leq 1
\]

(16)
We mention also that the speed of convergence increases if eq. (14) is replaced by the equation

$$W_{n+1}(\omega) = \begin{cases} H(\omega) & \omega \in B \\ A_n H_n(\omega) & \omega \in B \end{cases} \tag{17}$$

where $A_n$ is a suitable factor.

3. Adaptive extrapolation

We now assume that $h(t)$ is different from zero not in the entire interval $(-T, T)$ but only in a subset $b$ of this interval. Thus, $h(t) = 0$ for every $t$ in the complement $\tilde{b}$ of $b$. In the problem under consideration, $b$ is a set consisting of a finite number of points $t_i$ [see eq. (3)]. If the set $b$ is known, then we determine the function $h_n(t)$ of the $n$th iteration such that

$$h_n(t) = \begin{cases} w_n(t) & t \in b \\ 0 & t \in \tilde{b} \end{cases} \tag{18}$$

This is a modified form of eq. (12) and it results in a reduction of the effect of noise and an increase in the speed of convergence.

In our problem, the set $b$ is not known. In fact, our objective is to find it. However, the information that it consists of a finite number of points can be used to reduce the size of the truncation interval. This is done as follows:

We start the iteration as in Sec. 2. As the iteration progresses, the function $w_n(t)$ takes significant values only in a subset of the interval $(-T, T)$ because $w_n(t)$ tends to a sum of impulses as $n \to \infty$. This set is denoted by $b_n$ and is so defined that

$$w_n(t) \geq \varepsilon_n \quad t \in b_n$$

$$w_n(t) < \varepsilon_n \quad t \in \tilde{b}_n \tag{19}$$

where $\varepsilon_n$ is a suitable threshold level. It is desirable that $\varepsilon_n$ be as large as possible subject to the condition that all points $t_i$ be included in the corresponding set $b_n$. We have found from a number of numerical calculations that a reasonable choice is the minimum

$$\varepsilon_n = \min[|w_{n-1}(t)|] \quad t \in b_{n-1} \tag{20}$$

of the signal $w_{n-1}(t)$ in the set $b_{n-1}$ of the preceding iteration step.

We next form the function $h_n(t)$ by truncating $w_n(t)$ in the complement $\tilde{b}_n$ (fig. 6):
Fig. 6  Adaptive reduction of truncation set; $\varepsilon_n$: threshold level.

\[
h_n(t) = \begin{cases} 
  w_n(t) & t \in b_n \\
  0 & t \in \overline{b}_n
\end{cases}
\]  \hspace{1cm} (21)

we compute its Fourier transform $H_n(\omega)$, and continue as in eq. (14). A nested sequence of sets $b_n$

\[
b_{n-1} \supseteq b_n \supseteq b_{n+1} \supseteq \ldots
\]

results which for reasonably small noise levels converges to the set of points $t_i$. If, however, some of the coefficients $c_j$ of $h(t)$ [see eq. (2)] are small, the corresponding points $t_j$ might not be in all the sets $b_n$. This depends on the level of noise but precise conditions cannot be easily established in general. We are in the process of determining sufficient conditions for the complete recovery of $h(t)$ for various special cases involving a small number of points $t_i$.

The following modification of the method results in a reduction of the effects of noise and round-off errors: the set $b_n$ does not necessarily change at each iteration step, i.e., it is possible that $b_{n-1} = b_n$. We continue the iteration until $b_n$ is a proper subset of $b_{n-1}$. When this is observed, we replace the signal $w_n(t)$ by the signal $w_1(t)$ of the first iteration step, and start again where now the interval $(-T, T)$ is replaced by the set $b_n$. This reduces the total energy of the noise, and it eliminates the accumulation of errors from the preceding steps.

4. Numerical illustration

We shall use the method to determine the location and the reflection coefficients of a synthetic medium consisting of six homogeneous layers. The corresponding impulse response is an impulse train consisting of six unequal impulses as in figure 7. The observed signal is the sum

\[
z(t) = y(t) + n(t)
\]  \hspace{1cm} (22)

shown in figure 8a. The component $y(t)$ is obtained by convolving $h(t)$ with the signal $x(t)$ of fig. 8b.
The component \( n(t) \) is white noise with uniform distribution, zero mean and standard derivation \( \sigma = 0.05 \). The signal \( x(t) \) is the output of a physical transducer (see also fig. 3), and it determines the scale of the time axis. All other signals are computer generated.

Our objective is to determine numerically \( h(t) \) in terms of \( z(t) \) and \( x(t) \). The computations are carried out digitally on a single-precision minicomputer using as sampling interval \( \delta = 0.1 \mu s \). It is assumed that the unknown impulses are in the
interval \((-T, T)\) where \(T = 4\mu s\). This interval is generally known from the physical description of the problem (outer surfaces of interrogated medium), or it can be determined from the observed data. In our case, the interval $2T$ contains 80 sample points. All Fourier transforms are computed with an FFT size $N = 256$. The total processing interval is, thus, $N\delta = 25.6\mu s$, that is, about three times the truncation interval $2T$. The interval $N\delta$ must be sufficiently large to avoid aliasing errors. The sampling interval $\delta$ determines the limit of resolution. The parameters $N$ and $\delta$ are not critical.

Since it is known that the segment of $z(t)$ outside the interval $(-T, T)$ is noise, it is truncated prior to any processing.

We shall first attempt to determine $h(t)$ by direct deconvolution. It will become evident from the computations that the results do not give an adequate estimate of $h(t)$. For this purpose, we compute the Fourier transforms $Z(\omega)$ and $X(\omega)$ of $z(t)$ and $x(t)$ respectively and form the ratio

$$H_a(\omega) = \frac{Z(\omega)}{X(\omega)} \quad (24)$$

(fig. 9a). The inverse transform $h_a(t)$ of $H_a(\omega)$ is the result of direct deconvolution (fig. 9b). The unknown impulses are not recognizable.

The effect of the noise is most pronounced outside the interval $(\omega_1, \omega_2)$. To reduce it, we truncate $H_a(\omega)$ outside this interval. The resulting function $W_1(\omega)$ and its inverse $w_1(t)$ are shown in figure 10. Again, $w_1(t)$ is not a satisfactory estimator of $h(t)$ because the frequencies outside the interval $(\omega_1, \omega_2)$ are eliminated.

To determine the missing frequencies of $H(\omega)$, we apply the iteration described in Sec. 3 starting with the function $W_1(\omega)$ of figure 10. The results of the iteration are shown in figure 11a for various values of $n$. In figure 11b, we show in detail the function $w_n(t)$ for $n = 40$. In figure 12, we plot the normalized mean-square error

$$e_n = \frac{1}{E} \sum_{k=0}^{255} [h(k\delta) - w_n(k\delta)]^2$$
$$E = \sum_{k=0}^{255} h^2(k\delta) \quad (25)$$

as a function of $n$. As we see from these results, the unknown $h(t)$ is essentially fully recovered for $n = 40$. In fact, the locations $t_i$ of the six impulses are located exactly (see figure 7).

We next repeat the iteration using as the window not the pulse $P(\omega)$ of eq. (15), but the function

$$P_1(\omega) = \frac{1}{k} |X(\omega)| \quad (26)$$
Fig. 9 Estimate of $h(t)$ obtained through direct deconvolution
(a) Ratio $H_a(\omega) = Z(\omega)/X(\omega)$ of spectra of $z(t)$ and $x(t)$
(b) Inverse transform $h_a(t)$ of $H_a(\omega)$.

Fig. 10 Estimate of $h(t)$ obtained through deconvolution and truncation
(a) Reliable segment $W_1(\omega)$ of $H_a(\omega)$
(b) Inverse transform $w_1(t)$ of $W_1(\omega)$

where $k$ equals the maximum of $|X(\omega)|$. This function is between zero and one and is chosen because it favors the frequencies for which $|X(\omega)|$ is large, i.e., the portion of the frequency axis where $H_a(\omega)$ is reliably determined [see eq. (24)]. The window $P_1(\omega)$, however, has the drawback that it distorts the component of $Z(\omega)$ due to the useful signal $y(t)$.

In figure 13a, we show the results of the iteration. The function $w_40(t)$ is shown in figure 13b and the normalized mean-square error $e_B$ in figure 12. As we see from those figures, the error is reduced rapidly at the first 15 steps, but it re-
Fig. 11 (a) Results of adaptive iteration with rectangular window for n = 8, 16, 24, 32, and 40. (b) Detailed plot of $w_n(t)$ for n = 40.

Fig. 12 Mean-square error

$$e_n = \frac{1}{N} \sum_{k=0}^{N-1} \left[ h(k\delta) - w_n(k\delta) \right]^2$$

a: With rectangular window. b: With window $P_1(\omega)$ proportional to $|X(\omega)|$
Fig. 13 (a) Results of adaptive iteration with window $P_n(\omega)$ for $n = 8, 16, 24, 32, 40$. (b) Detailed plot for $n = 40$.

remains essentially constant for $n > 20$. The reason is that in the early stages of the iteration, the distortion due to the noise is rapidly reduced, and what remains is the distortion of the useful component of $Z(\omega)$ due to the fact that $P_n(\omega)$ is not constant in truncation interval. It seems, therefore, that a combination of the two windows leads to better results: We start with the window $P_n(\omega)$ until the set $b_n$ of the iteration is only a small part of the interval $(-T, T)$. We then continue with the rectangular window of eq. (15).

We conclude with an observation concerning the meaning of the term "resolution". This term is used extensively in several areas, and it has been given various definitions. These definitions, however, lead to a measure of resolution that is of the order of the duration $d$ of the transmitted signal $x(t)$, or, equivalently, inversely proportional to the bandwidth of $x(t)$. (In optics, the resulting number is inversely proportional to the diameter of the aperture.) The measure so described is adequate if we rely on the direct observation of the data for the determination of the minimum spacing between impulses (point objects, in optics). It is not, however, useful if the data are subjected to elaborate processing. As we have seen, if the data contain no noise, then we can separate two or more arbitrarily close points no matter how large $d$ is. The resolution is limited only by the size of the sampling interval $\delta$. This is not so if noise is present; however, in this case also the limitation is not simply the size of $d$, but it depends on the level of the noise and on the form of the unknown signal $h(t)$.

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