27. IMAGE RESTORATION

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Abstract. An extrapolation method is presented for recovering an object from its diffraction limited image. The high frequency components of the object, lost due to the finite size of the aperture, are restored by a numerical iteration involving only FFT. The method is particularly effective for objects consisting of isolated points. Various illustrations show that the resulting resolution far exceeds the Rayleigh limit.

A modification of the method yields an algorithm for recovering a distant object obscured by low-frequency clutter. The noise is removed with suitable filtering and the lost frequency components of the object are recovered by extrapolation.

1. Spectral Extrapolation

We shall develop a method for restoring an object that is distorted by a diffraction limited imaging system. For simplicity, we assume that the object is one-dimensional and completely coherent. The results can be readily extended to two-dimensions and to incoherent objects.

In Fig. 1, we show a simplified schematic of the imaging system. We denote by $f(x)$ and $g(x)$ the amplitude of the object and of its image respectively, and by $F(u)$ and $G(u)$ their Fourier transforms. Assuming that the system is ideal, we have

$$G(u) = \begin{cases} F(u) & |u| < r \\ 0 & |u| > r \end{cases}$$

where $r$ is the radius of the aperture.


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FIGURE 1 DIFFRACTION-LIMITED IMAGING SYSTEM: OBJECT \( f(x) \), IMAGE \( g_0(x) \) AND FIELD \( F(u) \) ON FOURIER PLANE

We are given \( g_0(x) \) and our problem is to find \( f(x) \).

a. Iteration

We shall show that, if the object \( f(x) \) vanishes for \( |x| > a \), then it can be completely recovered from \( g_0(x) \) by numerical iteration (Refs. 1-3).

For this purpose, we compute, first the Fourier transform \( G_0(u) \) of \( g_0(x) \). Clearly, \( G_0(u) = F(u) \) for \( |u| < \tau \) and \( G_0(u) = 0 \) for \( |u| > \tau \).

We truncate \( g_0(x) \) forming the function

\[
\hat{f}_1(x) = \begin{cases} 
  g_0(x) & |x| < a \\
  0 & |x| > a 
\end{cases}
\]  

(2)

We find the Fourier transform \( F_1(u) \) of \( \hat{f}_1(x) \) and form the function

\[
G_1(u) = \begin{cases} 
  F_1(u) & |u| > \tau \\
  F(u) & |u| < \tau 
\end{cases}
\]  

(3)

We find the inverse transform \( g_1(x) \) of \( G_1(u) \). This completes the first step of the iteration (Fig. 2).

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FIGURE 2. FIRST ITERATION STARTING WITH KNOWN IMAGE \( g_0(x) \) AND ITS FOURIER TRANSFORM \( G_0(u) \)

The next step is formed by replacing in Eq. (2) the function \( g_0(x) \) by the function \( g_1(x) \) and proceeding as in the first step.

The nth step is similar: starting from \( g_{n-1}(x) \), we form the function \( g_n(x) \) as in Eq. (2), the function \( G_n(u) \) as in Eq. (3), and its inverse \( g_n(x) \).

We prove in Ref. 1 that, in the absence of noise, the iteration converges to the unknown object \( f(x) \). This is not the case if noise is present. However, even then the method can be used to enhance the distorted image. This is done by terminating the iteration at an appropriate step so as to minimize the overall error (Ref. 1).

b. Point objects. If additional information about the object is available, then the iteration speed increases and the noise problem can be relaxed. A case of particular interest in many applications involves objects consisting of distinct points (neighboring stars, for example). In this case, \( f(x) \) consists of sharp peaks at certain points and the problem is to find the location and the amplitude of these peaks. The number of points need not be known in advance although this knowledge improves the convergence of the iteration.

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Adaptive extrapolation. To determine \( f(x) \), we start the iteration as in Fig. 2. Since \( f(x) \) consists of impulses, the function \( g_{n}(x) \) exhibits maxima for sufficiently large \( n \). When this is observed, a threshold level is set and only the region of \( g_{n}(x) \) above that level is retained. The remaining part, below the threshold level, is replaced by zero yielding the function \( f_{n+1}(x) \) shown in Fig. 3. The iteration step is completed as before. The threshold level is set low at first and is raised as the iteration progresses. As we show in the following illustrations, the object \( f(x) \) can be recovered even in the presence of considerable noise.

![Figure 3](image)

**Figure 3** Truncation of \( g_{n}(x) \) below a threshold level yielding \( f_{n+1}(x) \)

Example 1. In Fig. 4, we show an object \( f(x) \) consisting of two points in a noisy background, and its image \( g_{0}(x) \). It is evident from the figure that the two points are completely merged. The results of the iteration are shown for \( n=4 \) and \( n=30 \). As we see, the amplitudes and locations of the two points are recovered for \( n=30 \).

Example 2. In Fig. 5, we show an object consisting of 6 points in a noisy background and its image \( g_{0}(x) \). At the 12th iteration, we observe 6 maxima but the amplitudes are not correct. The 6 points are essentially recovered at the 100th iteration.

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FIGURE 4 \( f(x) \): UNKNOWN OBJECT CONSISTING OF TWO POINTS CONTAMINATED BY NOISE, \( g_0(x) \): KNOWN IMAGE, \( g_4(x) \): RESULT OF 4TH ITERATION, \( g_{30}(x) \): RESULT OF 30TH ITERATION

Example 3. In Fig. 6, we show the application of the method for the recovery of a two-dimensional object consisting of two points. The image is shown in Fig. 6b and the results of the iteration for \( n=4 \) and \( n=24 \) in Fig. 6c and d. At the 24th iteration the object is recovered completely.

2. Clutter elimination and object restoration

An object \( f(x) \) seen through a noisy medium yields an image

\[
y(x) = f(x) + n(x)
\]

We assume that the noise component \( n(x) \) has only low-frequency components and we seek to determine \( f(x) \). We can eliminate the noise \( n(x) \) by high-pass filtering. This can be done optically by blocking the center portion of the aperture of the viewing system. The resulting image \( y(f) \) is free of noise but is distorted because it consists only of the high frequencies of \( f(x) \). To recover \( f(x) \) from

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\( w_0(x) \), we apply the preceding method suitably modified: Reversing the role of high and low frequencies, we can show that the iteration, starting from \( w_0(x) \) yields \( f(x) \). The details are omitted.

![Graphs showing signal processing](image)

**FIGURE 5** \( f(x) \): **UNKNOWN OBJECT CONSISTING OF 6 POINTS CONTAMINATED BY NOISE**, \( g(x) \): **KNOWN IMAGE**, \( g_{12}(x) \): **RESULT OF 12TH ITERATION**, \( g_{100}(x) \): **RESULT OF 100TH ITERATION**

Example 4. In Fig. 7, we show the unknown object \( f(x) \) and the date \( y(x) \) containing the noise \( n(x) \). Filtering of \( y(x) \) yields the signal \( w_0(x) \). As we see from the figure, the noise is eliminated but the signal \( f(x) \) is strongly distorted. To find \( f(x) \), we apply the iteration method. At the 10th step, we obtain the signal \( w_{10}(x) \) shown in Fig. 7. This is essentially identical to the unknown object \( f(x) \).

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FIGURE 6  (a) UNKNOWN OBJECT, (b) KNOWN IMAGE (c) 4TH ITERATION, (d) 34TH ITERATION

FIGURE 7  f(x): UNKNOWN OBJECT, y(x): IMAGE OF f(x) VIEWED THROUGH CLUTTER, w₁(x): FILTER OUTPUT, w₃(x): 3RD ITERATION, w₁₀(x): 10TH ITERATION
References

